Mathematics in the Human Journey

Introduction

Mathematics is the science of analogy. Our brains are hard-wired to recognize that some things are similar to other things, in some respects. Thus a flock of sheep is similar in some ways to a set of notches cut into a piece of wood, or a collection of pebbles dropped into a leather bag. The floor plan of a building can be similar, in some ways, to a shape drawn on a piece of paper, or scratched in the sand with a stick. The ways in which things are similar eventually become things in their own right, and can develop a life of their own—a mathematical life. Mathematical knowledge consists of knowing what is preserved by such analogies (the invariants of a certain transformation) or, equivalently, what transformations will preserve certain properties. A collection of pebbles can be divided up and recombined in the same way that a flock of sheep can, except perhaps more easily. Pebbles are not woolly, and they do not bleat, but for certain operations they will suffice. A diagram of a building has the advantage that it can be erased and redrawn. It does not usually represent accurately the size of the building, but it does preserve the proportions, angles and shape. The use of such analogies paves the way for symbolic thinking, in which one thing, the symbol, is substituted for another, the thing being symbolized. Symbols allow us to construct models of reality that can record accurately the current state of affairs. But to the mathematician such models are dynamic: he knows what transformations can be made which will preserve the correspondence between the model and the thing being modeled. Thus a mathematician is able to manipulate his models of reality in ways that can be used to predict future outcomes. This is what gives him power over his world.

A distinction must be drawn between the mathematics possessed by the majority of adults in a community, which is required for the activities of daily life and which may become such an integral part of their way of thinking that they are unaware of it, and the mathematics possessed and deliberately cultivated by an elite, the results of whose insights may be of tremendous benefit to the community as a whole even though the insights themselves cannot be easily communicated. Thus a man packing the trunk of his car for a weekend trip is using counting or one-to-one correspondence ("days = pairs of socks?") and 3-D geometrical transformations ("Will this suitcase fit better if I turn it around?") without thinking that he is "doing mathematics." Another man, designing a computer chip, is using mathematical concepts of recent origin and unknown to most of us. (We can perhaps add to this hierarchy a "middle class" that is educated in the use of more sophisticated mathematical methods while not really understanding why they work.) Anyone who has ever taught mathematics, at any level, will know the difficulty of communicating its insights to others. It is more convenient just to give the results, accompanied perhaps by an invented story to justify them. The Egyptian priests who could forecast accurately the flooding of the Nile basin might not have been able to explain their method even it they had wanted to. Moreover, the source of esoteric knowledge can be protected deliberately in order to maintain a power relationship. It is perhaps not surprising then that mathematics, for much of its history, has been linked with mysticism and religion. Nevertheless, it would be wrong to suppose that only an elite is capable of making its own mathematics. The communication of mathematical ideas requires that the learner take the same steps that humanity as a whole has taken: the role of the teacher is to guide and accelerate this process. The child learning to count is not merely imitating or following a set of instructions, but discovering for itself the concepts of cardinality, order and grouping. It is now generally accepted among mathematics educators that everyone is capable of inventing their own mathematics, and indeed they must do so. In this sense we are all mathematicians, and we always have been.

Mathematics in Prehistory

The earliest evidence of mathematical activity is often taken to be the patterns of notches cut onto bones excavated from late Paleolithic sites, dating back to 35000 B.C. Some of these seem to be straightforward tallies; others have been interpreted as records of the passage of time, bearing a remarkable similarity to the calendar sticks still in use by Bushmen clans in Namibia. Tallies, which are still used to this day, and not only in primitive communities, keep a record of the number of days passed, the number of animals owned or the amount of a debt, but do not by themselves constitute a number system. Their use depends on the concept of a one-to-one correspondence between the marks made and the objects of interest (days, sheep, dollars). Number is the invariant of this transformation.

The ability to count, and the accompanying concept of number, evolved slowly from humble beginnings. The contemporary study of extant primitive peoples reveals to us some of the stages along the way, as does the study of comparative linguistics. First comes recognition of the difference between one, two and many: a distinction that survives in some ancient languages that have different forms for singular, dual and plural. Next is the gradual abstraction of the concept "two" until it is independent of the objects being described, as evidenced by the gradual replacement of object-specific number words by general-purpose ones. The mathematical definition of "two" as "the set of all sets with two elements" is not an empty play on words, but a formal statement of this abstraction process. The move from "two" to "three" may have taken many centuries: there are many context-specific words for "two": "pair," "brace," "couple": but not for "three." As life became more communal, organized and complex, the need for finer distinctions within the "many" arose. Early number words were often representatives of their class: wings, clover, legs, hand. The one-to-one correspondence between number words and a collection of objects mirrors the correspondence between tally marks and the objects tallied, but the use of an ordered sequence of number words or symbols in place of tally marks allows the cardinality of the set to be described by the last word, or symbol, in the sequence. Once this link between the ordinal and the cardinal was established, counting had begun.

Even some of the earliest evidence of tallying indicates a level of mathematical processing beyond merely one-to-one correspondence. A wolf's radius dated to 30000 B.C. has the tally marks clearly grouped into 5s. This grouping has an important role to play in counting. First of all, it makes tally marks recognizable number symbols. For example:

| | | |
| | | | is easily recognizable as 8, whereas | | | | | | | | is not.

It has been noted that humans, when shown a collection of objects, can say immediately, without counting, how many objects are in the collection up to about four objects, unless the objects are arranged or grouped in some way. Beyond four, we have to count the objects. This is reflected in some languages that have forms for singular, dual, triple, quadruple, and plural. Some hints of this can even be found in Latin. Such grouping is typical of the early number symbols used by the Egyptians and Sumerians around 3000 B.C. Chinese "rod numerals" also show this kind of grouping. These were used by mathematicians and scientists—the "everyday" system for writing numbers was quite different—and formed the basis for doing calculations and for the Chinese abacus. The earliest known use was "only" 200 B.C., but they probably go back much farther.

Secondly, grouping facilitates the development of a number system in which groups are themselves counted, and then groups of groups, this process being extended as far as necessary. Our own number system uses groups of ten: thus 324 represents 4 units, 2 groups of ten and 3 groups of ten groups of ten. Although ten was common in ancient times, and has triumphed as the (almost) universal group size, grouping by twos, fours, fives, twelves, twenties and sixties have all been used. There is evidence for some of these in modern number words: consider, for example, the French "quatre-vingts" or "four-twenties." The Sumerians had a highly sophisticated system based on tens and sixties, which remained in use in astronomical work throughout the Middle Ages, and still exists in modern units of time and angle. In contrast, some primitive tribes today count in the following way: "one, two, two-one, two-two, two-two-one, two-two-two."

Thirdly, grouping marks the beginning of true calculation, as opposed to merely "counting on." Consider the problem of determining a total for two groups with tally marks | | | | | | | | and | | | | | | . A counter will start counting the first group, then continue on through the second. If, however, the tallies are grouped: 卌| and 卌||| : then the result is easily seen to be 卌卌||||. Grouping allows us to hold and manipulate numbers in our minds, and to see relationships between them. Such manipulations were made concrete in the earliest calculating devices, the counting-board and the abacus.

Closely related to counting is the activity of measurement. Measurement becomes counting through the use of a standard unit (of length, time, weight). An early grouping activity, as shown by the Ishango bone (c. 8000 B.C.) discovered in Zaire, was the grouping of days into larger units of time, corresponding to phases of the moon. In this way, early man began to map and measure out his world, relating the large and unknown to the small and familiar, and thereby expanding the scope of his models of reality. Different standard units are useful for different purposes: a finger-joint for small lengths, a stride for longer ones. The groupings used for measurements, whereby a certain number of a smaller unit constitutes a larger unit, would need to be based on the practicalities of measurement. This is reflected in the wide variety of group sizes used (twelve inches in a foot, sixteen ounces in a pound), and may in part explain the variety of bases used in early number systems. It is not known to what extent such units were standardized in early societies, but there is some evidence that the henges, or stone circles, found in various parts of Northwestern Europe and dated to the late Neolithic or early bronze age, appear to have been constructed based on a standard unit of length, the Megalithic Yard. Among the artifacts found in the cities of the Indus Valley culture of India-Pakistan (Harappa, Mohenjo-Daro) are objects for measuring weights and lengths. In particular, one of the "rulers" that has been found is marked off in units of 1.32 inches, which has been called the “Indus inch.” The kiln-fired bricks they used for building have dimensions that are integer multiples of the “Indus inch.” Measurement also opens up the possibility of dividing up a basic unit into smaller units, leading to the concept of fractions. The grouping of days into months, based on observation of the moon, eventually led to conflict with the grouping of days into years, based on observation of the sun and stars, because the solar year is not an exact multiple of the lunar month. This conflict led to a great deal of mathematical activity in the production of calendars attempting to reconcile the two.

The abstraction of shape, like the abstraction of number, may have taken many thousands of years to develop. The shaping of tools, whereby an abstract shape carried in the mind is transferred back into the physical world, represents an early and highly significant achievement. The mathematical ability to manipulate shape in the mind, together with an appreciation of the relationship between form and function, allowed the development of new tools without the need for trial-and-error or discovery. Recognition of the shapes formed by the fixed stars provided a reference point for navigation as well as a calendar. Primitive analogies between these astral shapes and terrestrial objects persist to this day in the signs of the zodiac. Appreciation of shape also allowed our ancestors to “see,” and therefore to create, representative art, as in the paintings which adorn many Paleolithic caves.

Early Urban Civilizations

When Neolithic man began to settle in villages, cultivate crops and domesticate animals, the possibilities for using these basic mathematical abstractions of number, quantity and shape increased dramatically. Dwellings had to be designed and constructed, herds counted and crops shared out. From around 10000 B.C., at sites throughout the Near East, a system of symbolic counting tokens developed. These tokens were made of clay, up to one inch in size, and of various geometrical shapes, such as cone, sphere, cylinder or disk. It is thought that the shapes were used as a code to represent different items (animals, crops) and each stood for a specific quantity. Some represented different quantities of the same good: a cone for a small measure of grain, a sphere for a large measure. Their systematic use in the counting, sharing and exchange of goods provided an impetus, and a cognitive aid, to the development of number systems. Unlike static tally marks, groups of tokens can be rearranged, split into smaller groups, exchanged for larger measures. Physical rearrangement leads to mental rearrangement and the experience of invariant properties that hold true independently of the type of token. That the token system was administered by some central authority is suggested by the early standardization of the shapes and the use of an official seal to provide secure storage of groups of tokens, perhaps relating to a debt or contract. Thus it became necessary for individuals to adopt and adapt to a common system of symbols in order to partake in the economic life of the community. A common system allows, even forces, the communication of mathematical ideas and the development of an algorithmic approach to their implementation.

As community life became ever more complex, with the development of city-based civilizations in Mesopotamia, Egypt, the Indus Valley and the Yellow River from around 3000 B.C., mathematical knowledge became a vital tool in commerce, in building and in administration. Occupational and regional specialization lead to the growth of trade, which required knowledge of arithmetic for keeping accounts, exchanging merchandise, and computing simple and compound interest. Large-scale communal building projects such as irrigation systems, granaries, temples and city walls required geometrical knowledge for their planning and construction. Arithmetical calculations were also required for the number of bricks needed, the number of men to employ, the number of days it would take and the provisions necessary. Taxes had to be calculated, harvests shared out, and areas of land subdivided for cultivation. A reliable civic calendar was required for aligning communal activity with the cycles of nature, based on carefully observed and recorded astronomical data. In these early civilizations such mathematical work (recording, calculating, planning) was the preserve of a powerful elite, often the rulers, priests and scribes. In the story of the goddess Inana's descent to the underworld in Sumerian literature, she takes with her a measuring rod and measuring line, symbols of her power. In the Egyptian Book of the Dead, a king boasts of his ability to count using his fingers.

The increasing complexity of economic life was reflected in the increasing complexity of the token system, both in terms of the numbers required and the diversity of shapes. A total of about 500 different types have been found in the Near East region. A system of keeping groups of tokens in clay envelopes developed. In order to be able to check the contents without having to break the seal, marks were made on the outside of each envelope. Thus the tokens, themselves a symbol of something concrete in the real world, were replaced by something even more abstract, a set of symbolic marks on a clay tablet. One can imagine that this move to a higher level of generalization and abstraction would have had the effect that it still has today on learners of mathematics when they put away the wooden blocks and bead frames and start making marks on pieces of paper. An increase in efficiency is paid for by an increased reliance on rules and rote learning. Nevertheless, the invention of writing led to an explosion in the development and use of number systems.

Early Babylonian Mathematics

Our knowledge of Babylonian mathematics derives from the discovery and careful deciphering of thousands of clay tablets dating back to 2000 B.C., after the Akkadians had taken over Mesopotamia and adapted the earlier Sumerian culture. These tablets were written in cuneiform script, a set of wedge-shaped marks made in the clay by a triangular, cross-sectional stylus. Some are records of financial transactions, while others are apparently textbook exercises from the scribal schools. Some are quite sophisticated sets of mathematical tables: reciprocals, squares, and even cube root and trigonometric tables. The number system used has symbols for 1 and 10, and a place-value system using a base of 60. For example:

 would mean 2(60) + 10 + 4 = 134.

With this system, algebraic and geometric problems were solved: the division of a quantity of food among several people, the calculation of the area of a field or the volume of a granary, even number problems equivalent to the solution of quadratic equations. Some problems involved the use of a special symbol for the "unknown," like our ubiquitous "x." The Babylonian mathematicians did not, however, have an algebraic notation capable of expressing a general algorithm for solving a particular type of problem. Worked examples showed how problems of the same type could be solved, but with no explanation or justification of the method used. Although the problems always had an apparent practical context, it seems that the context was sometimes merely there to enable an interesting algebraic problem to be posed. Consider this example from the Hammurabi dynasty c. 1700 B.C.:

Length, width. I have multiplied length and width, thus obtaining the area. Then I added to the area, the excess of the length over the width: 183. Moreover, I have added length and width: 27. Required length, width and area.

We would express this problem as:

$$xy + x - y = 183$$
$$x + y = 27$$

Such problems impart the flavour of an interest in mathematics for its own sake, and not just for the solution of commercial and administrative problems. They imply the existence of an intellectual elite for whom mathematics has become a playground, a garden of delights. The phenomena of the world have been reduced to numbers and mathematical operations, and these have predictive power: we can calculate how many bricks are required to complete the building, how much seed is needed to plant the field. But these numbers and operations are found to have a rich life of their own, for those who understand them. The later Babylonians of the Assyrian and Seleucid periods (from 700 B.C.) applied their arithmetic to the heavens. By calculating first and second differences of the observed positions of the planets, and extrapolating or interpolating these differences, they were able to predict their daily positions. Given that the planets were variously worshipped as gods, one must speculate on the effect of this discovery that even they obeyed the rules of arithmetic.

Early Egyptian Mathematics

There is less direct evidence of the achievements of Egyptian mathematics. The papyrus on which they wrote proved less enduring than the baked clay tablets of the Babylonians. Two important papyri that have survived are the Rhind or Ahmes papyrus and the Moscow papyrus, both written in hieratic script by scribes working for the state and dating from about 1700 BC. The former famously promises the reader "knowledge of all obscure secrets," but, like the latter, it contains arithmetical problems and their solution. The number system used in hieroglyphic script was based on grouping in tens. There was no use of place-value, so each power of ten needed an additional symbol. Thus 1,325 would be written, reading from right to left, as:

In the later hieratic script, groups of symbols evolved into individual symbols. Thus hieratic numerals had different symbols for the integers from 1 to 9, but also a completely different set of symbols for 10, 20 to 90. The important step of using place-value was missed, which is one reason why the Egyptian number system was more cumbersome than that of the Babylonians. The other was their treatment of fractions. With few exceptions, fractions had a numerator of one, denoted by a bar symbol above the numerator, for example $\overline{III}$ for 1/3.

Other fractions had to be expressed as a sum of these simple fractions. This was very inconvenient for calculation, compared to the Babylonian system, which just extended the base-60 system into fractional units, as our decimal system does with base 10. Judging from the problems solved in the Rhind and Moscow papyri, the Egyptians used their mathematics in the administration of the state: determining wages, assessing taxes and dividing up land by area. They knew how to calculate the areas and volumes of geometrical shapes, and had a fairly accurate value for π. A typical problem from the Rhind papyrus is:

> Find the volume of a cylindrical granary of diameter 9 and height 10.

The solution is given as a set of instructions with no explanation. It begins by reducing the diameter by one-eighth and squaring the remainder, which implies a value for π of − . Solutions are set out in such a way that similar problems can be solved by following the same steps—this is clearly the intention. Thus, the examples are a way of giving algorithms for solving certain kinds of problems, in the absence of an algebraic notation. An analogy has been noted between such problems, allowing a higher level of abstraction: the problem type. The transformation here is the substitution of different numbers; the next problem asks for the volume when the diameter is 10. The invariant of this transformation is the steps taken in the solution. Thus, an infinity of different problems on the volumes of cylinders has become one problem and the solution is the algorithm, shown by example. Some of the algorithms seem a little crude: multiplication, for example, is achieved by a method of repeated addition. Some equation-type problems are solved by a "method of false position," in which an initial trial answer is taken and then adjusted to give the correct one. It is supposed that the comparative richness of Babylonian mathematics is the result of their greater exposure to social and political upheavals, and therefore to outside influences. Egypt was geographically isolated. Its relative stability might, however, have aided in the accumulation of astronomical observations and the accompanying production of calendars. Egypt used a fairly accurate calendar of 12 months of 30 days each, with five additional feast days at the end of the year. The adoption of this calendar has been dated to the year 4241 BC. The division of the 12 constellations of the zodiac into 36 "decans" was used for telling the time at night and led ultimately to the division of a day into 24 hours. The most important event in the calendar was the flooding of the Nile basin, which was heralded by the appearance in the morning sky of the star Sirius. Over a period of many years, the 365-day civic calendar lost its synchronicity with this event. It is claimed that the Egyptian priests knew, but kept secret, a more accurate estimate of the solar year, which they used to predict the date of the flood.

Early Indian Mathematics

At about the same time that urban centers were appearing in Mesopotamia and Egypt (ca. 3500 BCE), cities were also developing in the Indus River Valley of the Indian subcontinent (mostly in what is now Pakistan) and in the Yellow River Valley of China. Two problems limit our knowledge about the earliest civilizations in those places. In the case of the Harappan culture of the Indus Valley (named for Harappa, the first major city to be discovered), the written language of the culture has yet to be deciphered. In the case of China, in an attempt to rewrite history, an emperor in 213 BCE had all scholarly documents (and a number of scholars) destroyed. Nevertheless, even without written documentation, quite a bit is known about the early mathematics in these cultures.

The archeological discoveries in the Indus Valley cover an enormous area in and near the valley of the Indus River and its tributaries (1.2 million square kilometers) making it the largest of the ancient civilizations. The size, diversity, and complexity of its buildings and other artifacts suggest a complex culture that would have required an advanced mathematics comparable to that of Mesopotamia or Egypt – arithmetic for business and administration and geometry for construction projects. Among the artifacts are a number of "measuring instruments." There are sets of weights with ratios of 1, 2, 5, 10, 20, 50, etc., suggesting a decimal system of measurement. There are also "rulers" for measuring lengths, marked off in extraordinarily precise increments. The units of length vary somewhat from city to city, but in each location the kiln-fired bricks used in construction have dimensions that are integer multiples of the local unit of length. The unit of about 1.32 inches (33.5 mm) used in the city of Mohenjo-Daro has become known as the "Indus inch." A larger unit used from ancient times in northwestern India, the "gaz," is 25 Indus inches. The "gaz" is also very nearly identical to (0.36 inches longer than) the "Megalithic yard" used in western Europe.

The Indus Valley civilization disappeared by about 1500 BCE, around the time that the Aryan people from the northwest began to settle in India. The Aryans were the authors of the *Vedas*, the earliest literature of India of which we still have evidence (dating from about 1700 BCE to 500 BCE or later). Associated with the *Vedas* are "technical appendices" called *Vedangas,* which cover phonetics, grammar, etymology, verse, astronomy, and rules for performing rituals. The rules for rituals (*Kalpasutras*) include the *Sulbasutras*, composed mostly between 800 and 200 BCE, containing detailed instructions for altar construction. These instructions contain many mathematical procedures, especially using geometry, for constructing altars of various shapes but with the same area, or for finding the dimensions of an altar of a particular shape but twice the area. These procedures involve such things as Pythagoras' Theorem, several different approximations to the value of pi, and a very good approximation to the square root of 2 (577/408 = 1.414216, correct to 5 decimal places). The emphasis is on practical, applied mathematics. There is no attempt to prove any of the results or even to distinguish between exact and approximate results.

Interest in sacrifices and altars and their geometry waned with the appearance of two new religions, Buddhism and Jainism, in the mid-6th century BCE. These religions eventually gained the support of wealthy landowners and merchants, and the Jains became leading financiers. About the same time, Cyrus the Great of Persia and his successors created the largest empire the world had yet seen, an empire that included what is now western Pakistan. At this time Mesopotamian astronomical knowledge made its way to India. Slightly later, Alexander the Great invaded India (327 BCE), strengthening contacts with the West. Consequently, Indian mathematics of the Jaina period (ca 500 BCE to ca 400 CE) focused on business and astronomical applications with emphasis on arithmetic and algebra (solving equations). Number theory (see below) also received attention. Geometry, the main focus of the earlier Vedic period, received much less attention.

An enduring feature of Indian culture is a fascination with huge numbers. These numbers were used to describe the vast stretches of time and space in Vedic, Buddhist, and Jain cosmology. As far back as the Indus Valley civilizations, a base-ten number system had developed. Huge numbers were accommodated by giving a different name (actually several different names, depending on the time and place) to each power of ten, similar to the way we use the terms ten, hundred, thousand, million, billion, etc., today. In (American) English the largest commonly used power of ten is trillion or 10^{12}. In the modern metric system, there are prefixes for expressing large (and small) quantities: kilo- for 1000, mega- for one million, etc. Currently the largest official metric prefix is yotta- for 10^{24}, but in current practice tera- (10^{12} or one trillion) is the largest one is likely to encounter. The Yajurveda (one of the four major Indian Vedas) also gives names for each power of ten up to 1012. However, the famous epic, the Ramayana, in counting the sizes of the opposing armies, uses names for powers up to 10^{62}. And in the Lalitavistara Sutra, a collection of ancient Buddhist stories written down between 1 and 300 CE, there is a tale of Buddha being examined by a mathematician in which Buddha names powers of ten as high as 10^{421}! Jaina authors continued this fascination in connection with their cosmology, which envisioned limitless time and space. For measuring time, they had a unit of one purvis, which is 750 x 10^{11} days; and another unit, the shirsa prahelika, consisted of $8{,}400{,}000^{28}$ purvis, or about 5.7 x 10^{207} days! In contrast, the largest number for which the ancient Greeks had a name was the myriad (10,000), and the largest number for which there is a Chinese character is also 10,000. In the Egyptian hieroglyphic system, the largest number for which there was a symbol was one million. Some have speculated that the need to express such large numbers concisely was an impetus for the invention of the place value system in India.

In keeping with their interest in vast amounts of time and space, the Jaina mathematicians also developed a rather elaborate treatment of infinity—they had five different kinds of infinity. Their idea that not all infinities are equal was not accepted in Europe until Georg Cantor's work in the late 1800s. The Jains also understood the law of exponents ($x^a x^b = x^{a+b}$), and they studied problems involving numbers of permutations and combinations and arithmetic progressions. These topics can all be related to their interest in huge numbers.

In all the early Indian mathematics texts, rules and procedures are stated without proof. An important exception to this is the Bakhshali manuscript, thought to be from the Jaina period but of disputed age (dated as early as the 2nd century CE and as late as the 12th century). Besides including proofs and demonstrations, this manuscript contains a number of other advances. These include an accurate approximation to the square root of any integer (earlier works gave roots of specific numbers), the earliest Indian treatment of indeterminate equations (equations having more than one solution), the use of a symbol (a dot) to represent the unknown in an equation, and the use of a place-value number system that included zero.

The Indian Place-Value Number System with Zero

Because of the importance of the "discovery" of zero in the development of mathematics, and because Europe got its modern place-value number system from India (via the Islamic world), it may be useful to trace the development of Indian numerals and their use of zero. The oldest written numbers in India are from the Harappan (Indus Valley) civilization; but since its writing has yet to be deciphered, little can be said about its number system except that it was probably a decimal-based system (as mentioned above). The oldest deciphered written numbers appear in edicts carved on rocks and columns during the reign if Asoka (273-237 BCE), the third ruler of the Maurya dynasty. The numerals used (and the alphabetic characters) varied from place to place. Greek and Aramaean were used in Afghanistan. Kharoshthi (or Aramaeo-Indian) numerals were used in northern India. These appeared as early as the 4th century BCE and looked very much like Roman numerals, using symbols I and X (representing 1 and 4!) additively. Brahmi numerals were used elsewhere in India, and many later styles of written numerals (including the modern Western ones) are derived from Brahmi. In Brahmi, I, II, and III were used for 1, 2, 3, but 4-9 each had its own unique symbol, some of which look vaguely like modern Western numerals. There were additional symbols for 10, 20, 30, ... 100, 200, etc. These symbols were used additively, not in a place-value system. By the 5th century CE, Brahmi had evolved into the Gupta numerals (named for the Gupta dynasty, ca. 240-535 CE) from which evolved the Nagari ("urban") numerals, also called Devanagari—the Nagari of the Gods. When Islamic scholars later translated Indian texts, they adapted the Nagari numerals to suit their own style of writing. They referred to them as Hindi or Indian numerals, not (as we do) as Arabic numerals! In the eastern part of the Islamic empire, these numerals developed into the numerals used in modern Arabic. In the west, a different version became the "Ghubar" ('dust') numerals that eventually found their way into Europe to become our modern Western numerals.

As just mentioned, the Indian numerals initially were used additively. The use of a place-value system in India is first encountered not with written numbers but with words. Indian astronomer-mathematicians developed a system for expressing numbers in which each of the numbers 1-9, as well zero, was represented by a written (or spoken) word. Actually, many different words might be used for a given digit, allowing for some flexibility and creativity when stringing words together to form a large number. Thus, "1" might be indicated by the usual word for "one" or by "beginning," "earth," "moon," etc. "2" might be "two," "pair," "eyes," "arms," etc. Such words were strung together into one long word using a place-value system. For example, today we might express 1075 by saying "one thousand seventy-five" or even "ten seventy-five," but to be perfectly clear, we might say "one-zero-seven-five." This is how the Indians formed their word numbers—except that they used no punctuation, so it would have been written more like "onezerosevenfive." The oldest known example of such word numbers is in a Jaina cosmology text (the *Lokavibhaga* or "Parts of the Universe"), which is dated by the writer (when translated into our Julian calendar) as Monday, August 25th, 458 CE.

The oldest documented occurrence of Indian numerals, rather than words, being used in a place-value system with zero is on inscriptions found in Malaysia and Cambodia, using numerals derived from Brahmi, dated 683 CE. In India itself, the oldest undisputed examples are from inscriptions in stone (thus not likely to be forgeries) using Nagari numerals at Gwalior dated 875-876. Other examples with earlier dates (as early as the late 6th century CE) are found on inscriptions in copper recording donations by kings and wealthy persons to the Brahmans, but some scholars claim that these are later forgeries. Also in the 6th century (662 CE), a Syrian bishop named Severus Sebokt, irked by the arrogance of those who attributed all knowledge to Greek civilization, wrote about the Indians' incomparable methods of calculation using only nine figures. Though there is no mention of zero, the mention of using only nine figures makes it clear that knowledge of a decimal-based place-value system from India was known at this time.

Of course, a place-value system with a zero existed much earlier. Babylonian scientists had a base-60 place-value system as far back as the 19th century BCE. This system was used both for whole numbers and for fractions (other early civilizations treated fractions differently from whole numbers), but it lacked a "decimal point," so the value of each "place" was determined by context—just as today we know that "seven ninety-nine" means 7.99 when talking about the price of a sandwich, but it means 799 when it refers to the price of a personal computer. Initially, the system also lacked a zero; an empty "place" was simply represented by a space. It was not until around the 3rd century BCE (the Seleucid or Hellenistic period) that a symbol (two wedges) for an "empty place" began to be used. It served only as a placeholder; it was not treated as a number (i.e., as something that could participate in mathematical operations like addition, subtraction, and multiplication). In mathematical texts, this placeholder "zero" appears only in the middle of a number, never at the beginning or end of the number. So the value of each "place" was still determined by context. But in some tablets of astronomical data it appears in final position and also in initial position. In both these cases it represented the "ones" place. When it appeared in initial position, it meant that the subsequent digits were fractional digits. Greek astronomers (but *not* Greek mathematicians) began using this system in the 2nd century BCE, but only for fractions (with zero as the initial digit). They used their own alphabetic numerals instead of Babylonian numbers, and their symbol for the placeholder zero was a small circle. Subsequently Arab astronomers acquired the system from the Greeks, and they passed it on as the base-60 "degrees-minutes-seconds" system that is still used in astronomy. Scholarly opinions differ as to whether the modern symbol for zero derives from that small circle used by the Greek astronomers or whether the Indian mathematicians independently developed a similar symbol for their zero. In any case, the Indian mathematicians seem to have been the first to treat zero as a number rather than just a placeholder.

Early Chinese Mathematics

The oldest mathematical artifacts from China are the "oracle bones" from the time of the Shang dynasty (about 1500 to 1000 BC). These "bones" (actually tortoise shells as well as flat cattle bones) were used for divinations by soothsayers to Shang nobles. They contain writing that includes numerals. Though some of the characters used have subsequently changed, both the writing and the numerals are essentially those of modern China. Numbers were represented in a decimal system using symbols for the digits from 1 to 9 and powers of ten: 10, 100, 1000, etc. They were written in the same way as they were spoken, which is more or less the same way that we say numbers. Thus for example the number 234 would be written as "2 hundred 3 ten 4" using the corresponding Chinese symbols.

In fact, the Chinese, in addition to the standard system of writing numbers described above, also had a place-value system. As noted above, the standard system is a written version of the spoken numbers. The place-value system, called the rod numerals, is a written version of how the Chinese did their calculations. Like many other ancient cultures (Babylonians, Egyptians, Greeks, Romans), the Chinese used their written numbers primarily to record quantities, not to do arithmetic—in much the same way that we use Roman numerals even today. Calculations were done with bamboo rods on a counting table—an early form of abacus. (The familiar Chinese "rod abacus" that is still used today did not appear in China until much later, around the 15th century AD.) The oldest written treatise on calculation with rods dates from the 5th century AD, but artifacts (such as coins) displaying rod numerals appear much earlier, and some authorities date the rod numerals to 1000-500 BC. The numerals 1 to 9 were represented using one to five bamboo rods in various orientations. The rods were arranged on the counting table, which consisted of a rectangular grid. Each row represented a number and each column represented a "place" (a power of ten) within the number. By using several rows, along with appropriate manipulations of the rods, it was possible to do addition, subtraction, multiplication, division, powers and square roots. The written "rod numerals" were simply the written representation of the numbers as they appeared on the counting table. The one missing piece (until around 900 AD) was that there was no written symbol for zero. This was not a problem when using the counting table itself; an empty cell represented zero. But when written down without a symbol for zero, it was easy to confuse, say, 23 and 203 (which would have been written 2 3 with extra space to indicate the missing numeral).

Other early cultures used similar forms of the "abacus." In some cases pebbles were used rather than bamboo rods; in other cases written numbers were used rather than rods or pebbles. The grid itself might be produced in dust or sand on the ground or on a table, or using a wax tablet (the first really portable "calculator" used by the Romans).

Greek and Roman Mathematics

With the emergence of the classical Greek civilization from around 600 BC came the first named mathematicians. Thales (ca 600 BC) and Pythagoras (ca 500 BC) were both reputed to have traveled widely, in Egypt and Babylonia, and to have brought back mathematical knowledge to Greece. Later came Plato, Aristotle, Euclid, Archimedes and a host of other names familiar to the modern mathematician. No original manuscripts of their work survive, and our knowledge of their work and lives is dependent on second-hand sources admixed with legend. Nevertheless, the scale of their accomplishments in mathematics is clear. Some of the underlying tools of mathematics—abstraction, generalization and logic—were explicitly recognized and pursued. The concept of "proof" was explored, leading to an axiomatic structure in which certain basic, self-evident premises or "axioms" are shown, following clearly defined rules of logic, to lead to new knowledge. This could have lead to a dry and mechanical formalism, but it was applied to an area of mathematics where intuition and dynamic imagery were natural tools: geometry. The results are enshrined in the most famous and influential of mathematics textbooks, Euclid's Elements. Starting with 10 supposedly self-evident postulates or axioms, such as "Things equal to the same thing are equal to each other," the books build up a structure of deductions or "theorems" describing the nature of space.

This combination of intuition with formal logic proved to be a potent mixture for investigating and structuring the world. Our intuition encourages us to make great leaps forward, perhaps seeing similarities or analogies in unexpected places. But sometimes intuition leads us astray, and the analogy breaks down on more careful inspection. To be useful, the mental structures by which we represent the world, and which we manipulate when "thinking," need to maintain a kind of self-consistency.

The Greek mathematicians and philosophers investigated the nature of this self-consistency.

Greek mathematics was seen as divided into: arithmetic, geometry, mechanics, astronomy, optics, surveying, musical harmony and calculation. Though the more philosophically inclined, such as Plato, reputedly scorned the "corruption" of mathematics by its application to problems in the real world, regarding the purpose of mechanics, for example, as merely to illustrate geometrical theorems, this did not deter other mathematicians like Archimedes from constructing mechanical devices for military and civilian use. Thus, mathematics for the Greeks was both a thing of beauty, to be admired in its own right, and a powerful tool for investigating and shaping the world. The use of basic mathematics in commerce and administration has now become, dare one say, axiomatic.

Despite the great advances in mathematics made by the Greeks, their number system lacked the "place-value" concept used by the Babylonians and in modern decimal notation. Their early system of numerals was similar to the familiar Roman system, with different symbols for each power of ten and additional symbols for groups of five. Each symbol was then repeated as many times as was necessary. For example 236 would be written as HH_____, where H is hundred (hekaton), _ is ten (deka) and _ is five (penta). Calculation with these numerals was very difficult, so counting boards were used. Gradually a new system emerged, in which the 27 letters of the Greek alphabet were used, in order, to represent the integers from 1 to 9, then powers of ten from 10 to 90, then hundreds from 100 to 900. Thus 236 in this system was ___. In manuscripts a convention, such as a horizontal stroke, was used to distinguish numerals from words. It is clear that this system was a deliberate invention designed to improve on the old system. It first appeared in the 5th century BC but was not adopted officially until the 1st century BC. Its great advantage was that it allowed complex calculations to be made in writing, without the need for an abacus or counting board.

The Romans are credited with no new developments in mathematics, and indeed are blamed by some mathematical historians for precipitating, by their pragmatic attitude towards knowledge, a mathematical "Dark Ages" lasting from the demise of Greek civilization until the peak of the Islamic one. Nevertheless, there is clear evidence that they were great users of mathematics. Their construction of aqueducts, roads and bridges, the organization of their army and the administration of their vast empire all testify to their assimilation and widespread use of the mathematics of earlier cultures. The Romans were great organizers, and organizing is a mathematical activity. Julius Caesar employed a Greek mathematician to organize the calendar so that it would not drift away from the solar year. His suggestion of a 365-day year with an extra day every fourth year was adopted in 45 BC.

Ancient to Modern

When the Arabs established their Islamic Empire around 700 AD, they inherited, translated, synthesized and extended the mathematical works of the Babylonians, Indians and Greeks. An important contribution was their popularizing of the use of Indian "Hindu-Arabic" numerals. Arab mathematicians like Omar Khayyam developed "algebraic thinking"—the formal manipulation of arithmetical relationships involving unknown quantities. They did not use algebraic symbols, but expressed the manipulations in terms of sequences of arithmetical operations, or "algorithms"—from the Arabic mathematician Al-Khwarizmi (whose book "Hisab al-jabr w'al-muqabala" gave us the term algebra). Islamic calendars and prayer rituals demanded accurate knowledge of astronomy and geography—the five daily prayers were prescribed at times determined by the position of the sun, facing in the direction of Mecca. This led to new work on the accuracy of trigonometric tables, and further development of the astrolabe, an instrument for fixing one's position accurately and an important tool for navigation. Contact between the Moslem Empire and mediaeval Europe led eventually to the translation into Latin of the lost Greek mathematics together with the later contributions of the Arabs and Indians.

The gradual development of the modern, scientific view of the world has seen mathematics play a key role. This development could be characterized as the conflict between Perceived and Believed Reality. In the 13th century AD the "Merton School," based at Merton College, Oxford—Grossetest, Roger Bacon, Ockham and others—began the exploration of physical phenomena, such as rainbows, based on careful observation, and the construction of mathematical models that attempted to explain them as simply as possible. This paradigm of Science as Observation plus Mathematical Modeling made mathematics the tool and language of science. In Art, too, the emphasis shifted to a requirement to represent figures, landscapes and scenes as they appear to the eye, rather than as they are known to be. The development of Perspective in art in the 15th to early 16th century AD, by artists such as Piero, Da Vince and Dürer, made use of mathematical principles from the science of Optics. Mathematical rules and devices for achieving correct perspective led eventually to the theory of projective geometry (the study of properties preserved by projection onto any flat surface – the invariants of the projection transformation), and the technique of drawing by cross-wires, an early coordinate system.

The Renaissance in the 16th century saw an increase in trade, especially international trade, which led to demand for accurate accounting, navigation and coping with different scales of weights and measures. The need for mathematicians was met by sons of merchants and craftsmen, taught in trade schools, guilds and workshops, which emphasized the teaching of practical mathematics. The invention of the printing press enabled the publication and wide distribution of textbooks in practical mathematics, and tracts emphasizing its importance and usefulness. The prestige of mathematics grew, owing to its contribution to the wealth of nation states. The English statesman Francis Bacon, in the early 17th century, emphasized its importance as a resource for wealth creation.

The scientific requirement for models to closely fit observations made in the physical world, rather than conforming to an idealized inner reality, also found expression in astronomy. The Alexandrian Greek astronomer Ptolemy had constructed a geometric model of the universe with the earth as centre and the sun and planets in circular orbits. Circles were the ideal—hence the correct—shape. To conform to the growing body of accurate data, the centres of these circular orbits were themselves required to move in circles. Gradually, in a series of steps taken by Copernicus, Galileo and Kepler, and against considerable opposition from the establishment, the geocentric model with circular orbits was replaced by a heliocentric one with ellipses. Newton's laws of gravitation and motion completed the new model by giving it a mathematical underpinning: the perfect circle was replaced by the straight line, the inverse square law and the notion of continuous change.

The apparent paradoxes involved in dealing with continuous change and the infinite were expressed by the Greek philosopher Zeno around 500 BC. His best-known example is Achilles and the Tortoise: Achilles is unable to overtake the tortoise because whenever he reaches the point where it was, it has moved somewhere else. The paradox is created by dividing a finite amount of time, the time taken to overtake the tortoise, into an infinite number of smaller amounts. The infinite and the infinitesimal arose in problems of quadrature, or calculating the area of curved shapes, and gradient, or calculating the instantaneous direction of a curved line. Related practical problems were the calculation of volumes and the location of centers of gravity, both important in architecture and shipbuilding. Archimedes developed a method of calculating area using a "limiting argument." For example, the area of a circle must lie between the areas of two polygons, one fitting inside the circle and one fitting outside. By increasing the number of sides of the polygons, one obtains an increasingly accurate estimate, and the true area is obtained as the limit when the number of sides becomes infinite. Such procedures became increasingly common in the 17th century in the work of Kepler, Cavalieri and Fermat, culminating in the "invention" of calculus by Newton and Liebnitz. Since many scientific principles involve rates of change, the importance and power of calculus for describing dynamic phenomena cannot be overstated.

The usefulness of mathematics in representing physical laws precisely and concisely was boosted by the development of symbolic algebra, which went hand-in-hand with the increasing use of Hindu-Arabic numerals during the 16th and 17th centuries. Descartes, in 1637, developed a method for representing geometrical constructions algebraically, so that curves could be described by equations, thus reversing (or completing?) a centuries-old trend for algebraic problems to be solved geometrically. If letters of the alphabet could be used to represent unknown quantities, they could also be used to represent the concepts related by scientific theories. The modern expression of Newton's 2nd Law of Motion, "F = ma" is a concise representation of the relationship between the force acting on a body, its mass and the resulting acceleration. It relates not only the concepts, but also the quantities involved; in fact, it gives a precise definition for the concept of "force" and a unit for measuring it. This procedure whereby aspects of the physical world are mapped onto letters of the alphabet, which are manipulated according to the formal rules of algebra and then mapped back onto the physical world, has led to an explosion in the power of predictive science. A recurring phenomenon in the part played by mathematics in the advancement of science is the extent to which "Pure" mathematics, developed for its own sake by mathematicians unconcerned with possible applications, has later become "Applied" mathematics, a useful tool for solving problems in the physical world. The search in the 16th century for algorithms for solving cubic and higher-order equations led to knowledge of algebraic manipulation vital to the physical sciences; the introduction of imaginary numbers, involving the square root of minus one, as a purely synthetic device for solving equations has eventually found application in electronics, and abstract number theory is now used in creating (and cracking) encryption codes. The requirement for a mathematical description and evaluation of knowledge has become all-pervasive. In the fields of Economics, Psychology and even Sociology, mathematical expressions are sought for laws governing human behaviour.

Computation has long been an important aspect of mathematics. In the 16th century, calculation algorithms using Hindu-Arabic numerals began to supplant the use of the Roman abacus, although in some cultures, particularly China and Japan, the abacus has remained popular. The invention of logarithms by Napier in 1614, to ease the computational burden of complex astronomical calculations, enabled approximate answers to be obtained quickly in situations, like navigation, where speed was important. The 20th century has seen electronic computers revolutionize the field of computation. Mathematical developments in binary arithmetic and Boolean algebra have been at the forefront of this revolution. The need for individuals possessing computational skills has disappeared, to be replaced by the need for computer programmers with skills in algorithmic thinking: the dismantling of complex tasks into a structured hierarchy of components.

Probabilistic and statistical thinking are relatively recent developments, beginning in the mid-17th century with the mathematical investigation of games of chance, particularly dice, by Pascal and Fermat. In the early 18th century Bernoulli and De Moivre made clearer the relationship between the theoretical probability measuring the likelihood of an event and the long-run proportion of times it occurs in repeated trials. This mathematical description of the patterns formed by random events found a use in the pricing of annuities and insurance policies: individual deaths are unpredictable but death rates for large populations tend to be stable. Later, in the early 19th century, Legendre and Gauss invented and justified the "least squares" method for estimating the orbit of an asteroid based on error-prone observations. This approach allows for measurement and formulation errors in scientific laws: instead of being expected to give exact answers, a scientific theory can be regarded as an attempt to account for the most important factors influencing an observable phenomenon, with all other factors lumped together as "experimental error." In 1900 Pearson invented the chi-squared test for comparing observed data with the predictions from scientific theory, allowing random variation. In the 1930s Fisher developed mathematical methods for dividing up the observed variation in a phenomenon into "explained" and "unexplained" components. This led to a new approach in designing experiments to learn about the influence of controllable factors in the presence of many uncontrollable ones, allowing mathematically expressed scientific theories to be developed in areas like Biology and Economics, where there is considerable subject-to-subject variation.

While mathematics continues to play a vital role in the organization of modern societies, as well as in technological innovation and scientific research, there are signs that the supply of people trained in the mathematical sciences may be drying up. The general populace regards mathematics as both boring and difficult. Business leaders and politicians like to boast of their lack of achievement in mathematics. Mathematics is not a popular subject at school, and many university departments are struggling to fill positions. In the business world, mathematical expertise is a commodity to be bought and sold, as and when it is required. The electronic calculator and the computer have removed the need for the traditional mathematical techniques taught in school. The challenge now is for us to develop a clearer understanding of what mathematics really is, to find more effective ways of teaching the real underlying mathematical skills and ways of thinking to a larger section of the population, and to promote mathematics, not only for its importance for the welfare of society, but for its contribution to the enlargement of our inner lives.

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